PROJECTIONS OF 3-POLYTOPES*

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ABSTRACT

Given any 3-dimensional convex polytope P, and any simple circuit C in the 1-skeleton of P, there is a convex polytope P' combinatorially equivalent to P, and a direction such that if P' is projected orthogonally in this direction, then the inverse image of the boundary of the projection is the circuit in P' corresponding to the circuit C in P.

1. Introduction. This paper is another investigation of the question of how the combinatorial structure of a convex 3-dimensional polytope (hereafter to be called a 3-*polytope*) is related to its shape (see [1, 3, 6, ch. 13]).

If P is a 3-polytope, then by a *regular projection* of P we mean an orthogonal projection of P onto a plane in a direction not parallel to any face of P. We shall prove the following theorem about such projections:

THEOREM 1. *If S is a simple circuit in the graph of a 3-polytope P, then there is a 3-polytope P' of the same combinatorial type as P, such that the corresponding circuit in P' is the inverse image of the boundary of some regular projection of P'.*

2. Definitions. The *graph* of a 3-polytope P is the graph formed by the edges and vertices of P. A graph is said to be 3-polyhedral of it is isomorphic to the graph of some 3-polytope. A theorem of Steinitz [2, 7, 8] states that a graph (without multiple edges) is 3-polyhedral if and only if it is planar and 3-connected. Using this fact we may assume that all 3-polyhedral graphs are embedded in the plane. If G is a graph emvbedded in the plane π and R is a connected component of $\pi \sim G$, bounded by a circuit S of G, then $R \cup G$ is called a *face* of G. A facet of a 3-polytope P is a 2-dimensional face of P, thus the faces of the graph of P correspond in a natural way to the facets of P.

Two 3-polytopes P and Q are *combinatorially equivalent* if there is a 1-1 incidence preserving function of the faces (of all dimensions ≤ 3) of P onto the

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faces of Q. By a theorem of Grünbaum and Motzkin [4, p. 498] two 3-polytopes are combinatorially equivalent if and only if their graphs are isomorphic.

3. Proof of Theorem 1. Our proof uses a method of proof used for Steinitz' Theorem [2, 6, 8]. The idea of the proof is to take a planar 3-connected graph G, find an edge e of G such that $G \sim e$ is homeomorphic to some planar 3-connected graph G' (such an edge will be called a *removable edge* of G). By induction there is a 3-polytope P' whose graph is G'. We add a line segment L across a facet F of P' so that the graph consisting of G' and L is isomorphic to G. We "bend" the face F along L and then adjust, one at a time, the vertices of P' and the planes determined by the facets of P' so that the vertices and facets match up as they did in P' except that F is split into two facets. This creates a polytope P whose graph is G. This process will be referred to as *splitting* F or as *facet splitting*. Of particular importance to us is the fact that this bending and adjusting can be done so that each facet and vertex is moved an arbitrarily small amount (see [2] for details).

The proof of Theorem 1 is by induction on the number of edges of P. The case where there are only six edges in P is easily verified. We consider two cases:

Case I. S is a circuit of length 3 and each vertex of S is joined to a common vertex v.

In this case we choose three planes supporting P on the three edges of S. If the three planes do not meet at a point then we may project parallel to three planes. If they meet at a point p we take a projective transformation of P which sends p to infinity and then the three planes will be taken onto planes which have no common point, and again we have a direction in which we may project.

Case II. S is not as described in Case I. We choose an edge e not on S such that $G \sim e$ is homeomorphic to a planar 3-connected graph G' (Lemma 2 will show that this can be done).

By induction there is a 3-polytope P' whose graph is G, with a circuit S' (corresponding to the circuit S in P) and a regular projection π such that S' projects onto the boundary of $\pi(P')$. We now choose a line segment L across a face F of P', as in the proof of Steinitz' theorem, bend F and adjust planes and vertices to obtain a polytope combinatorially equivalent to P. If we make the bend and the adjustments small enough then S will still be projected onto the boundary of $\pi(P)$. To see this choose one point in the relative interior of each edge of P' that is not on S and also choose such a point on L. We can choose neighborhoods of these points which project onto the interior of $\pi(P')$. If the bending and adjustments are small enough then there are neighborhoods of these points which will still project into the interior of $\pi(P)$ thus the only edges which can project onto the boundary are the edges of S.

We are left with the task of proving that the edge e exists.

LEMMA 1. *Let G be a planar 3-connected graph and let S be a simple circuit in G. There is a subgraph H of G, containing S, that is homeomorphic to the graph of the tetrahedron.*

PROOF. Let v_1 be a vertex of S. Since G is 3-connected there is a path Γ_1 from v_1 to some other vertex v_2 of S with no other vertices of S on Γ_1. Let v_3 be another vertex of S and let Γ_2 be the path from v_1 to v_2 along S, which contains v_3. If every path from v_3 returns to Γ_2 before it meets Γ_1 or $S \sim \Gamma_2$ then the graph can be disconnected by removing v_1 and v_2 which contradicts the 3-connectedness of G. If we add a path Γ_3 from v_3 to some vertex of Γ_1 or $S \sim \Gamma_3$ then we have constructed the desired subgraph of G.

LEMMA. 2. *Let G be a planar 3-connected graph that is not the graph of the tetrahedron and let S be a simple circuit in G. If S is not a circuit of length three with each vertex joined to a common vertex then there is a removable edge that does not lie on S.*

PROOF. We shall construct a sequence of subgraphs $G_1, G_2, \cdots, G_n$ of G such that G_1 is homeomorphic to T, the graph of the tetrahedron, each G_i is homeomorphic to some 3-polyhedral graph, and G_n is obtained from G_{n-1} by adding an edge across a face of G_{n-1} dividing the face into two faces (such a process will be called *face splitting.*)

By Lemma 1 there is a subgraph of G homeomorphic to T containing S. Let G_1 be such a subgraph with a maximum number of vertices. Proceeding by induction, suppose we have constructed G_{i-1}. In our construction we now consider two cases.

Case I. There exists a 2-valent vertex v_1 in G_{i-1}. Let Γ_1 be the arc (i.e. maximal path in G_{i-1} with all interior vertices 2-valent in G_{i-1}) containing v. From at least one 2-valent vertex v_2 of Γ_1 there must be a path Γ_2 in G which meets some other arc before it meets Γ_1, for otherwise G could be disconnected by removing the endpoints of Γ_1.

We let G_i be the graph obtained from G_{i-1} by adding to G_{i-1} the longest such path, Γ_3 from v_2. If G_{i-1} is homeomorphic to a 3-polyhedral graph H then G_i

is homeomorphic to a graph obtained from H by splitting a face. If $G_i = G_n$ the path must be an edge.

Case II. G_{i-1} has no 2-valent vertices. Let v_1 be a vertex of G_{i-1} which meets an edge e of G that is not an edge of G_{i-1}. Let Γ_1 be a path beginning with e, with the property that only its endpoints meet G_{i-1}. If Γ_1 does not end at a vertex v_2 where v_1 and v_2 are endpoints of an edge (note every arc is an edge in Case II) then adding Γ_1 would correspond to a face splitting. If it does end at such a vertex v_2 then the edge v_1v_2 is an edge of S, for otherwise we could replace v_1v_2 by Γ_1 which implies that either we did not add a maximal path at some step or we did not choose a maximal subgraph homeomorphic to T.

If v_1v_2 is an edge of S we observe that Γ_1 contains 2-valent vertices because G does not contain multiple edges. From one of these vertices v_3 there must be a path Γ_2 in G which meets $G_{i-1} \cup \Gamma_1$ only at the endpoints of Γ_2 and has only one endpoint on Γ_1, for otherwise we could disconnect G be removing the endpoints of Γ_1.

Let the other endpoint of Γ_2 be v_4. Let Γ_3 be the path from v_1 to v_3 along Γ_1 and from v_3 to v_4 along Γ_2. By the above argument, if we cannot add Γ_3 to G_{i-1} then the edge v_1v_3 is an edge of S.

Let Γ_4 be the path from v_2 to v_3 along Γ_1 and from v_3 to v_4 along Γ_2. Again, either we can add Γ_4 to G_{i-1} or v_2v_3 is an edge of S. Thus if we cannot add one of these three paths then S is a circuit of length 3. We now add $\bigcup_{i=1}^{3} \Gamma_i$ to G_{i-1}. We may do this as long as G_i is not G_n. But if it were then S would be a circuit of length three consisting of the neighbors of v_3 which is a contradiction.

We now have constructed the sequence $G_1, \cdots, G_n$ and we see that in constructing G_n we have added an edge that is not on S.

The search for generalizations to higher dimensions is not very fruitful. The following examples show that the two most obvious generalizations to 4-polytopes are not true.

EXAMPLE 1. In [4], Grünbaum and Sreedharan show that there exists a simplicial 4-polytope P and a subcomplex S of its boundary complex that is a topological 3-cell with the property that for any polytope P' combinatorially equivalent to P there is no point in 4-space from which all points of S' (the corresponding subcomplex in P') are visible. This shows that there is no polytope P' combinatorially equivalent to P such that the boundary of S' projects onto the boundary of $\pi(P')$ for some projection π. In other words our theorem does not generalize to subcomplexes of the boundary complexes of 4-polytopes that are topological 2-spheres.

EXAMPLE 2. We show that the theorem does not generalize to circuits in 4-polytopes by giving an example of a 4-polytope P with a circuit C such that no regular projection π of any polytope combinatorially equivalent to P will project the circuit onto the boundary of $\pi(P)$. Let P be the cartesian product of a square and a hexagon. There is a subcomplex S of the boundary complex of P that is topologically a torus, namely, the product of the boundary of the square with the boundary of the hexagon. If we label the vertices of the square 1, 2, 3, 4, and the vertices of the hexagon $1,2,\cdots,6$, then we may indicate the vertices of S by pairs of integers. Using this notation we shall describe a circuit C on S which is knotted (in the boundary of P) by giving the sequence of vertices of C:

$$\begin{array}{l}(1,1)\ (1,2),\ (4,2),\ (4,3),\ (3,3),\ (3,4),\ (2,4)\\ (2,5),\ (1,5),\ (1,6),\ (4,6),\ (4,1),\ (3,1),\ (3,2)\\ (2,2),\ (2,3),\ (1,3),\ (1,4),\ (4,4),\ (4,5),\ (3,5)\\ (3,6),\ (2,6),\ (2,1),\ (1,1).\end{array}$$

Since C is knotted it cannot lie in any subcomplex of P that is topologically a 2-sphere. Since the inverse image of a boundary of a regular projection of P onto E^3 is a 2-sphere we see that C cannot project onto the boundary of any such regular projection.

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